Journal of Sound and Vibration (2002) **253**(3), 669–685
doi:10.1006/jsvi.2001.4070, available online at http://www.idealibrary.com on IDEAL®

AN EXACT METHOD FOR THE FREE VIBRATION ANALYSIS OF TIMOSHENKO–KELVIN BEAMS WITH OSCILLATORS

J. CABAŃSKI

Department of Applied Mechanics, University of Technology and Agriculture, Faculty of Mechanical Engineering, Kaliskiego str.7, 85-791 Bydgoszcz, Poland

(*Received* 17 *July* 2000, *and in revised form* 3 *September* 2001)

In this paper, a modern exact method is proposed for solving the problem of free vibrations of a Timoshenko-type viscoelastic beam with discrete rigid bodies, connected to the beam by means of viscoelastic constraints. The phenomenon of free vibrations of this discrete–continuous system is described by a set of three partial and two subsystem ordinary differential equations with generalized boundary conditions and initial conditions. Vector notation of the equations allows one to identify the self-adjoint linear operators of inertia, stiffness and damping. In this case, these operators are not homothetic hence a separation of variables in this set of equations is possible only in a complex Hilbert space. Such separation of variables leads to ordinary differential equations of motion with respect to time as well as to a set of three ordinary differential equations with respect to a spatial variable and two subsystem algebraical equations. Solution of the boundary-value problem was carried out in the classical way, but its results are complex conjugated. Using these results and the fundamental principle, describing the orthogonality property of complex eigenvectors, the problem of free vibrations of the system with arbitrary initial conditions has been finally solved exactly.

1. INTRODUCTION

In conventional terminology mechanical systems are divided into discrete and continuous types. However, in practice combined systems, i.e., discrete–continuous, are also met. Such systems are much more complicated in mathematical description and dynamic response analysis than the isolated discrete or continuous system. Because of these complications, approximating methods of dynamic analysis of discrete–continuous systems are mostly used. The essence of approximation for the well-known finite element (FEM) analysis is discretization of the continuous subsystem which belongs to combined system. Another approximation method consists of expansion of particular solutions as series. Exact methods solving dynamic problems are well known but only for conservative examples and for some specific cases of non-conservative systems.

Bibliography describing classical approximation methods is very extensive and illustrative examples are included in references [1–7]. Despite the containing popularity of classical approximation methods, their inconvenience sometimes discouraged investigators from applying and evolving these methods. In the cases in which classical approximation methods are not satisfactory, one should apply the more exact methods wherever possible as shown by the continuous search for more accurate methods of vibrations analysis of compound systems (see, e.g., references [6–9]).

The aim of this paper is to introduce an exact method of solving the problem of free vibrations of viscoelastic Timoshenko beams in which concentrated rigid bodies are connected to the beam by means of viscoelastic constraints described by the Voigt–Kelvin model. Moreover at the beam ends, generalized boundary conditions are assumed. In order to realize the intended purpose of this paper, the fundamental principles given in reference [8] have been used.

2. FORMULATION OF THE PROBLEM

2.1. PHYSICAL MODEL

The investigations in this paper have been carried out on the physical model of the mechanical system shown in Figure 1. The continuous subsystem in this physical model is a viscoelastic Timoshenko beam with optional attachments at its ends. The rheological properties of the material of the beam are described by a Voigt–Kelvin model. The existing constraints at the ends of beam show, in general, all the practical attachment cases. The properties of the assumed model depend on the stiffness of the spiral springs and type of joints used for mounting them to the beam. The discrete subsystem consists of a set of translational and rotational oscillators. Each oscillator is a discrete rigid body with both translational and rotational inertia as well as viscoelastic constraints described by Voigt–Kelvin model. Mutual exclusion of any interaction between translatory and rotatory motion of the oscillator implies that each discrete rigid body can be presented as two independent rigid bodies, i.e., a particle and a thin disc (see Figure 1).

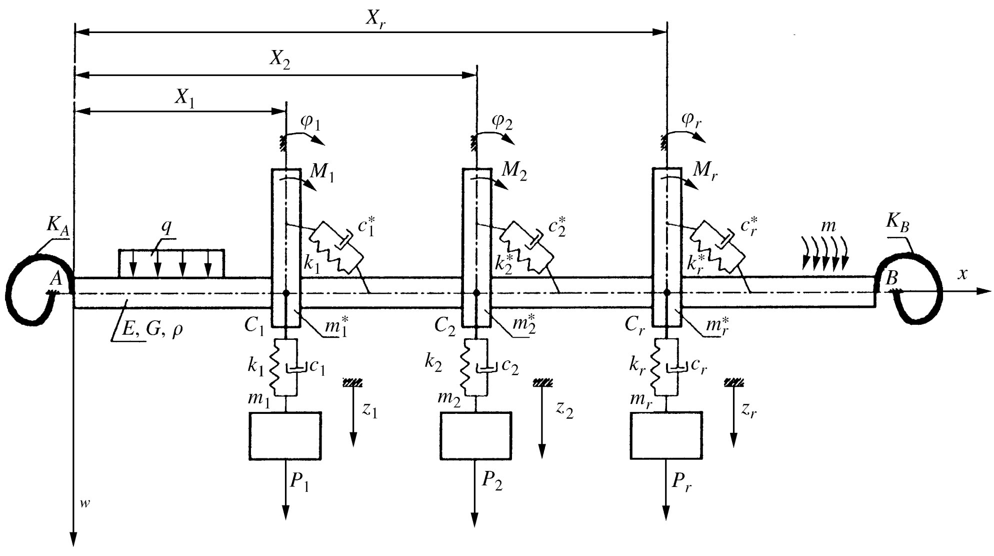

Figure 1. Physical model of Timoshenko–Kelvin beam with oscillators.

2.2. MATHEMATICAL MODEL

Vibrations of the mechanical system, presented in Figure 1 are described by the set of coupled differential equations:

$$\mu \frac{\partial^2 w}{\partial t^2} - \left(1 + \alpha \frac{\partial}{\partial t}\right) D \frac{\partial \gamma}{\partial x} + \sum_{j=1}^{r} \left(k_j + \frac{\partial}{\partial t} c_j\right)(w - z)\delta(x - x_j) = q,$$

$$\mu^* \frac{\partial^2 \psi}{\partial t^2} - \left(1 + \beta \frac{\partial}{\partial t}\right) R \frac{\partial^2 \psi}{\partial x^2} - \left(1 + \alpha \frac{\partial}{\partial t}\right) D\gamma + \sum_{j=1}^{r} \left(k_j^* + \frac{\partial}{\partial t} c_j^*\right)(\psi - \varphi)\delta(x - x_j) = m,$$

$$\frac{\partial w}{\partial x} - \psi - \gamma = 0, \quad \sum_{j=1}^{r} \left[m_j \frac{\partial^2 z}{\partial t^2} - \left(k_j + \frac{\partial}{\partial t} c_j\right)(w - z)\right]\delta(x - x_j) = \sum_{j=1}^{r} Q_j \delta(x - x_j), \quad (1)$$

$$\sum_{j=1}^{r} \left[m_j^* \frac{\partial^2 \varphi}{\partial t^2} - \left(k_j^* + \frac{\partial}{\partial t} c_j^*\right)(\psi - \varphi)\right]\delta(x - x_j) = \sum_{j=1}^{r} M_j \delta(x - x_j),$$

together with suitable boundary conditions

$$g_s(a, t) = 0, \quad s = 1, 2, \quad a = 0, l \quad (2)$$

and initial conditions

$$w_0 = w(x, 0), \quad \psi_0 = \psi(x, 0), \quad z_{0j} = z_j(0), \quad \varphi_{0j} = \varphi_j(0),$$

$$\dot{w}_0 = \partial w/\partial t|_{t=0}, \quad \dot{\psi}_0 = \partial \psi/\mathrm{d}t|_{t=0}, \quad \dot{z}_{0j} = \mathrm{d}z_j/\mathrm{d}t|_{t=0}, \quad \dot{\varphi}_{0j} = \partial \varphi_j/\partial t|_{t=0}, \quad (3)$$

where $D = k'GA$ and $R = EI$ are the shearing and flexural rigidities of the beam, respectively, $\mu = \rho A$ and $\mu^* = \rho I$ are the mass and the mass moment of inertia of the beam per unit length, respectively, k' is the shearing factor, and $\delta(\cdot)$ denotes the Dirac delta function. $z = z(x, t)$ and $\varphi = \varphi(x, t)$ denote auxiliary dummy functions which are filtered by the Dirac delta functions at the points x_j and simultaneously suppressed outside these points. Moreover, further parameters are defined as $w_j = w(x_j, t)$, $\psi_j = \psi(x_j, t)$, $z_j = (x_j, t)$ and $\varphi_j = \varphi(x_j, t)$. Equations (1) are a compilation of the equations in references [1, 2, 5, 9].

In order to express the boundary conditions (2) in explicit form, the equations of constraints at the beam ends can be applied as

$$g_s(x, t) = \sum_{p=1}^{2} \frac{\partial^k}{\partial x^k} \left[\alpha_{sp}^{(a)} w(x, t) + \beta_{sp}^{(a)} \psi(x, t)\right], \quad s = 1, 2, \quad (4)$$

where $\alpha_{sp}^{(a)}$ and $\beta_{sp}^{(a)}$ are the coefficients describing the types of beam attachments at its ends and $k = \delta_{2p}$, with δ_{2p} being a Kronecker delta. These coefficients are determined from consistency conditions of generalized internal forces and displacements of the beam and constraints at its ends. After differentiation of the right-hand side of equation (4) and replacing x by $a = 0, l$ successively, this formula becomes the fully explicit form.

It will be convenient for further investigations to rewrite equations (1) in the vector form [8]

$$\mathbf{M} \frac{\partial^2 \mathbf{u}}{\partial t^2} + \mathbf{L} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{K}\mathbf{u} = \mathbf{F}. \quad (5)$$

where $\mathbf{u} = [w, \psi, \gamma, z, \varphi]^{\mathrm{T}}$ and $\mathbf{F} = [q, m, 0, \sum_{j=1}^{r} Q_j \delta(x - x_j), \sum_{j=1}^{r} M_j \delta(x - x_j)]^{\mathrm{T}}$ are the vectors of displacements and loads of the system (see Figure 1) respectively. As well as

$$\mathbf{M} = \begin{bmatrix} \mu & 0 & 0 & 0 & 0 \\ 0 & \mu^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sum_{j=1}^{r} m_j \delta(x - x_j) & 0 \\ 0 & 0 & 0 & 0 & \sum_{j=1}^{r} m_j^* \delta(x - x_j) \end{bmatrix},$$

$$\mathbf{K} =$$
$$\begin{bmatrix} \sum_{j=1}^{r} k_j \delta(x - x_j) & 0 & D\dfrac{\partial}{\partial x} & -\sum_{j=1}^{r} k_j \delta(x - x_j) & 0 \\ 0 & \left[-R\dfrac{\partial^2}{\partial x^2} + \sum_{j=1}^{r} k_j^* \delta(x - x_j)\right] & -D & 0 & -\sum_{j=1}^{r} k_j^* \delta(x - x_j) \\ D\,\partial/\partial x & -D & -D & 0 & 0 \\ -\sum_{j=1}^{r} k_j \delta(x - x_j) & 0 & 0 & \sum_{j=1}^{r} k_j \delta(x - x_j) & 0 \\ 0 & -\sum_{j=1}^{r} k_j^* \delta(x - x_j) & 0 & 0 & \sum_{j=1}^{r} k_j^* \delta(x - x_j) \end{bmatrix},$$

$$\mathbf{L} =$$
$$\begin{bmatrix} \sum_{j=1}^{r} c_j \delta(x - x_j) & 0 & \alpha D\dfrac{\partial}{\partial x} & -\sum_{j=1}^{r} c_j (x - x_j) & 0 \\ 0 & \left[-\beta R\dfrac{\partial^2}{\partial x^2} + \sum_{j=1}^{r} c_j^* \delta(x - x_j)\right] & -\alpha D & 0 & -\sum_{j=1}^{r} c_j^* \delta(x - x_j) \\ \alpha D\,\partial/\partial x & -\alpha D & -\alpha D & 0 & 0 \\ -\sum_{j=1}^{r} c_j \delta(x - x_j) & 0 & 0 & \sum_{j=1}^{r} c_j \delta(x - x_j) & 0 \\ 0 & -\sum_{j=1}^{r} c_j^* \delta(x - x_j) & 0 & 0 & \sum_{j=1}^{r} c_j^* \delta(x - x_j) \end{bmatrix}$$

are the linear operators of inertia, stiffness and damping respectively. It is obvious that the above operators are self-adjoint because of their symmetry [8] but they are not homothetic.

Let one assume that operators $\mathbf{K}$ and $\mathbf{L}$ can be homothetic only in the case when the equalities $\beta = \alpha$, $c_j = \alpha k_j$ and $c_j^* = \alpha k_j^*$ are satisfied, this implying, that $\mathbf{L} = \alpha\mathbf{K}$, where α is a homothety factor and is not equal to zero.

3. BOUNDARY-VALUE PROBLEM

3.1. SEPARATION OF VARIABLES

In case of free vibrations, i.e., when $\mathbf{F} \equiv \mathbf{0}$, equation (5) reduces to the form

$$\mathbf{M}\frac{\partial^2 \mathbf{u}}{\partial t^2} + \mathbf{L}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{K}\mathbf{u} = \mathbf{0}. \tag{6}$$

The classical Fourier method of separation of the variables in equation (6) can be applied in cases when $\mathbf{L} \equiv \mathbf{0}$, i.e., for the undamped system or, if the operators $\mathbf{K}$ and $\mathbf{L}$ are homothetic when the vectors appearing in equation (6) are co-linear. In general, these vectors are co-planar and separation of the variables in equation (6) can be achieved only when the problem is extended to the complex Hilbert space. Such an extension makes it possible to separate the variables as in the classical case so that

$$\mathbf{u} = \mathbf{U}T, \tag{7}$$

where $\mathbf{U} = [W, \Psi, \Gamma, Z, \Phi]^{\mathrm{T}}$ is the vector of the complex modes of vibrations, $T = T(t)$ denotes a scalar function of motion, $W = W(x)$, $\Psi = \Psi(x)$, $\Gamma = \Gamma(x)$, $Z = Z(x)$ and $\Phi = \Phi(x)$, but remembering that Z and Φ play the role of auxiliary dummy functions.

Substituting equation (7) into equation (6) one obtains the ordinary differential equation of motion [8]

$$\dot{T} - \mathrm{i}\mathbf{v}T = 0, \tag{8}$$

and the vector equation of mechanical impedance [8]

$$(\mathbf{v}^2\mathbf{M} - \mathrm{i}\mathbf{v}\mathbf{L} - \mathbf{K})\mathbf{U} = \mathbf{0}, \tag{9}$$

where $\mathbf{v} = \mathrm{i}h + \omega$ stands for the complex frequency of vibration. Here ω and h are responsible for the oscillations and damping of motion respectively.

Equation (9) can be written in a scalar form expressed by a set of three ordinary differential and two subsystems of linear algebraic equations as

$$\mathbf{v}^2\mu W + (1 + \mathrm{i}\mathbf{v}\alpha)D\frac{\mathrm{d}\Gamma}{\mathrm{d}x} - \sum_{j=1}^{r}(k_j + \mathrm{i}\mathbf{v}c_j)(W_j - Z_j)\delta(x - x_j) = 0, \tag{10a}$$

$$\mathbf{v}^2\mu^*\Psi + (1 + \mathrm{i}\mathbf{v}\beta)R\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + (1 + \mathrm{i}\mathbf{v}\alpha)D\Gamma - \sum_{j=1}^{r}(k_j^* + \mathrm{i}\,\mathbf{v}c_j^*)(\Psi_j - \Phi_j)\delta(x - x_j) = 0, \tag{10b}$$

$$\frac{\mathrm{d}W}{\mathrm{d}x} - \Psi - \Gamma = 0, \quad j = 1, 2, \ldots, r, \tag{10c}$$

$$\mathbf{v}^2 m_j Z_j + (\kappa_j + \mathrm{i}\mathbf{v}c_j)(W_j - Z_j) = 0, \qquad \mathbf{v}^2 m_j^*\Phi_j + (\kappa_j^* + \mathrm{i}\mathbf{v}c_j^*)(\Psi_j - \Phi_j) = 0, \tag{10d, e}$$

subject to the new boundary conditions

$$G_s(a) = 0, \qquad s = 1, 2, \quad a = 0, l, \tag{11}$$

which, by analogy with equation (4), produces

$$G_s(x) = \sum_{p=1}^{2} \frac{\mathrm{d}^k}{\mathrm{d}x^k}\,[\alpha_{sp}^{(a)} W(x) + \beta_{sp}^{(a)} \Psi(x)], \quad s = 1, 2. \tag{12}$$

It should be noted that $W_j = W(x_j)$, $\Psi_j = \Psi(x_j)$, $Z_j = Z(x_j)$ and $\Phi_j = \Phi(x_j)$. The conditions governing equation (12) are the same as those for equation (4).

3.2. SOLUTION OF THE SET OF EQUATIONS

For solving equations (10), the classical procedure which is presented in, e.g., references [4, 5] has been applied. Hence, the general solutions of equations (10a–c), with the help of equations (10d, e) have the form

$$W = \sum_{s=1}^{2} A_s \sin\lambda_s x + B_s \cos\lambda_s x + \sum_{j=1}^{r} [\kappa_{\mathrm{I}j} W_j g_{11}(x, x_j) + \kappa_{\mathrm{I}j}^{*} \Psi_j g_{21}(x, x_j)]\mathrm{H}(x - x_j),$$

$$\Phi = \sum_{s=1}^{2} a_s A_s \cos\lambda_s x - a_s B_s \sin\lambda_s x + \sum_{j=1}^{r} [\kappa_{\mathrm{I}j} W_j g_{12}(x, x_j) + \kappa_{\mathrm{I}j}^{*} \Psi_j g_{22}(x, x_j)]\mathrm{H}(x - x_j),$$

$$\Gamma = \sum_{s=1}^{2} (\lambda_s - a_s)[A_s \cos\lambda_s x - B_s \sin\lambda_s x] + \sum_{j=1}^{r} \left\{\frac{\mathrm{d}}{\mathrm{d}x}[\kappa_{\mathrm{I}j} W_j g_{11}(x, x_j) + \kappa_{\mathrm{I}j}^{*} \Psi_j g_{21}(x, x_j)]\right.$$

$$\left. - [\kappa_{\mathrm{I}j} W_j g_{12}(x, x_j) + \kappa_{\mathrm{I}j}^{*} \Psi_j g_{22}(x, x_j)]\right\}\mathrm{H}(x - x_j), \tag{13}$$

where A_s and B_s, $(s = 1, 2)$, are arbitrary integration constants and $\mathrm{H}(x - x_j)$ is the Heaviside function. Equations (13) include the abbreviations

$$\lambda_s = \sqrt{\Lambda + (-)^{s-1}\sqrt{\Lambda^2 + \Pi}}, \quad a_s = ((1 + \mathrm{i}\mathbf{v}\alpha)D\lambda_s^2 - \mathbf{v}^2\mu)/(1 + \mathrm{i}\mathbf{v}\alpha)D\lambda_s \quad (s = 1, 2),$$

$$\Lambda = \frac{\mathbf{v}^2[(1 + \mathrm{i}\mathbf{v}\beta)\mu R + (1 + \mathrm{i}\mathbf{v}\alpha)\mu^* D]}{2(1 + \mathrm{i}\mathbf{v}\alpha)(1 + \mathrm{i}\mathbf{v}\beta)DR}, \quad \Pi = \frac{\mathbf{v}^2\mu[(1 + \mathrm{i}\mathbf{v}\alpha)D - \mathbf{v}^2\mu^*]}{(1 + \mathrm{i}\mathbf{v}\alpha)(1 + \mathrm{i}\mathbf{v}\beta)DR},$$

$$\kappa_{\mathrm{I}j} = \kappa_{\mathrm{II}j}\kappa_j/(\kappa_{\mathrm{II}j} - \kappa_j), \qquad \kappa_{\mathrm{I}j}^{*} = \kappa_{\mathrm{II}j}^{*}\kappa_j^{*}/(\kappa_{\mathrm{II}j}^{*} - \kappa_j^{*}),$$

$$\kappa_j = k_j + \mathrm{i}\mathbf{v}c_j, \quad \kappa_j^{*} = k_j^{*} + \mathrm{i}\mathbf{v}c_j^{*}, \qquad \kappa_{\mathrm{II}j} = \mathbf{v}^2 m_j, \quad \kappa_{\mathrm{II}j}^{*} = \mathbf{v}^2 m_j^{*}, \tag{14}$$

and the Green functions

$$g_{11}(x, x_j) = b_{11}[a_2 \sin\lambda_1(x - x_j) - a_1 \sin\lambda_2(x - x_j)],$$

$$g_{21}(x, x_j) = b_{21}[\cos\lambda_1(x - x_j) - \cos\lambda_2(x - x_j)],$$

$$g_{12}(x, x_j) = b_{12}[\cos\lambda_1(x - x_j) - \cos\lambda_2(x - x_j)],$$

$$g_{22}(x, x_j) = b_{22}[a_1 \sin\lambda_1(x - x_j) - a_2 \sin\lambda_2(x - x_j)], \tag{15}$$

where

$$b_{11} = (a_2\lambda_1 - a_1\lambda_2)^{-1}/(1 + \mathrm{i}\mathbf{v}\alpha)D, \quad b_{21} = a_1a_2(a_2\lambda_1 - a_1\lambda_2)^{-1}/(1 + \mathrm{i}\mathbf{v}\alpha)D,$$

$$b_{12} = -(a_1\lambda_1 - a_2\lambda_2)^{-1}/(1 + \mathrm{i}\mathbf{v}\beta)R, \qquad b_{22} = (a_1\lambda_1 - a_2\lambda_2)^{-1}/(1 + \mathrm{i}\mathbf{v}\beta)R. \quad (16)$$

To determine the quantities W_j and Ψ_j, the following recurrent formulas were derived directly from equations (13) as

$$W_j = \sum_{s=1}^{2} [A_s \sin\lambda_s x_j + B_s \cos\lambda_s x_j] + \sum_{k=1}^{j-1} [\kappa_{\mathrm{I}k} W_k g_{11}(x_j, x_k) + \kappa^*_{\mathrm{I}k} \Psi_k g_{21}(x_j, x_k)],$$

$$\Psi_j = \sum_{s=1}^{2} [a_s A_s \cos\lambda_s x_j - a_s B_s \sin\lambda_s x_j] + \sum_{k=1}^{j-1} [\kappa_{\mathrm{I}k} W_k g_{12}(x_j, x_k) + \kappa^*_{\mathrm{I}k} \Psi_k g_{22}(x_j, x_k)]$$

$$(j = 1, 2, \ldots, r), \quad (17)$$

where

$$g_{11}(x_j, x_k) = b_{11}[a_2 \sin\lambda_1(x_j - x_k) - a_1 \sin\lambda_2(x_j - x_k)],$$

$$g_{21}(x_j, x_k) = b_{21}[\cos\lambda_1(x_j - x_k) - \cos\lambda_2(x_j - x_k)],$$

$$g_{21}(x_j, x_k) = b_{21}[\cos\lambda_1(x_j - x_k) - \cos\lambda_2(x_j - x_k)],$$

$$g_{22}(x_j, x_k) = b_{22}[a_1 \sin\lambda_1(x_j - x_k) - a_2 \sin\lambda_2(x_j - x_k)]. \quad (18)$$

Alternatively, one can transform equations (10d, e) into

$$Z_j = -(\kappa_{\mathrm{I}j}/\kappa_{\mathrm{II}j}) W_j, \qquad \Phi_j = -(\kappa^*_{\mathrm{I}j}/\kappa^*_{\mathrm{II}j})\Psi_j. \quad (19)$$

Equations (19) imply an existence of the motionless nodes O_j and O^*_j at the translational and rotational components of the "complex constraints" of the jth oscillator, as shown in Figures 2(a) and 2(b) respectively.

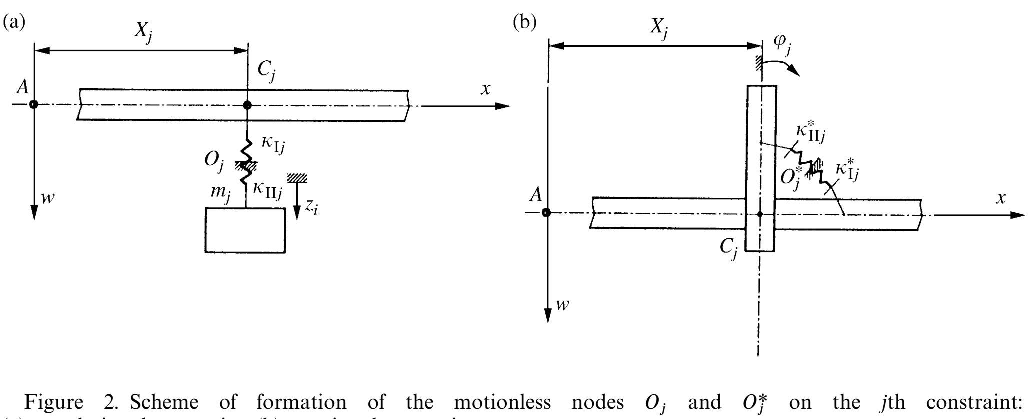

Figure 2. Scheme of formation of the motionless nodes O_j and O^*_j on the jth constraint: (a) translational constraint, (b) rotational constraint.

These motionless nodes split this mechanical system but yet do not exclude a dynamic interaction between the discrete rigid bodies and the continuous subsystem. Confirmation of the motionless nodes' existence can also be found in reference [10].

3.3. SOLUTION OF THE BOUNDARY-VALUE PROBLEM

By applying solution (13) to formula (12), then substituting the results into boundary conditions (11), the system of homogeneous linear algebraic equations can be constituted, which in a matrix notation takes the symbolic form

$$\mathbf{AX} = \mathbf{0}, \tag{20}$$

where $\mathbf{A}$ is a coefficient matrix with respect to $\boldsymbol{\nu}$, and $\mathbf{X}$ is the vector of unknowns of the equations system.

The system of equations has a non-trivial solution, providing that the matrix $\mathbf{A}$ is singular, i.e., the determinant of this matrix must vanish. Hence, the transcendental, complex frequency equation can be written in the symbolic form as

$$\det \mathbf{A} = 0. \tag{21}$$

The solution of equations (21) is presented as an infinite sequence of the complex eigenfrequencies $\{\boldsymbol{\nu}_n\}$, where $\boldsymbol{\nu}_n = \mathrm{i}h_n \pm \omega_n$, $(n = 1, 2, \dots,)$.

By considering the singularity of the matrix $\mathbf{A}(\boldsymbol{\nu}_n)$, the solution of equations (20) gives an infinite sequence of the complex vectors $\{\mathbf{X}_n\}$, here $\mathbf{X}_n = [A_{1n}, A_{2n}, B_{1n}, B_{2n}]^{\mathrm{T}}$ corresponding to $\boldsymbol{\nu}_n$. By substituting $\boldsymbol{\nu}_n$ and $\mathbf{X}_n$ ($n = 1, 2, \dots$, successively, into equations (13) and using the results from equations (19), one obtains an infinite sequence of the complex eigenvectors $\{\mathbf{U}_n\}$ of the boundary-value problem, here $\mathbf{U}_n = [W_n,\ \Psi_n,\ \Gamma_n,\ Z_{1n}, \dots, Z_{rn},\ \Phi_{1n}, \cdots, \Phi_{rn}]^{\mathrm{T}}$ corresponding to $\boldsymbol{\nu}_n$, and satisfying equations (10) and (11). Consequently, the eigenvector can also be written in compact form, i.e., $\mathbf{U}_n = [W_n,\ \Psi_n,\ \Gamma_n,\ Z_n,\ \Phi_n]^{\mathrm{T}}$.

This solution of the boundary-value problem and the fundamental principle [8]

$$([(\boldsymbol{\nu}_n + \boldsymbol{\nu}_m)\mathrm{i}\mathbf{M} + \mathbf{L}]\mathbf{U}_n, \mathbf{U}_m) = N_n \delta_{nm}, \tag{22}$$

describing the generalized orthogonality condition of the eigenvectors $\mathbf{U}_n$ and $\mathbf{U}_m$ with tapering operator $[(\boldsymbol{\nu}_n + \boldsymbol{\nu}_m)\mathrm{i}\mathbf{M} + \mathbf{L}]$, is the basis for solving the free and forced vibrations problems. Here δ_{nm} denotes the Kronecker delta.

4. FREE VIBRATIONS

The general solution of equation (6) with homogeneous boundary conditions (2) and initial conditions (3) is a linear combination of linearly independent particular solutions,

$$\mathbf{u} = \sum_{n=1}^{\infty} \mathbf{U}_n T_n. \tag{23}$$

Replacing T_n in equation (23) by the general solution of the differential equation (8), i.e., $C_n \exp(\mathrm{i}\boldsymbol{\nu}_n t)$ produces

$$\mathbf{u} = \sum_{n=1}^{\infty} C_n \mathbf{U}_n \exp(\mathrm{i}\boldsymbol{\nu}_n t), \tag{24}$$

where C_n are arbitrary integration constants.

The existence of technically permissible initial conditions (3) implies the existence of an expansion of the right-hand side of equation (24).

In order to find the constants C_n,

$$C_n = ([\mathbf{M}(\mathrm{i}\boldsymbol{\nu}_n \mathbf{u}_0 + \dot{\mathbf{u}}_0) + \mathbf{L}\mathbf{u}_0], \mathbf{U}_n)/([2\mathrm{i}\boldsymbol{\nu}_n \mathbf{M} + \mathbf{L}]\mathbf{U}_n, \mathbf{U}_n), \tag{25}$$

which owing to the orthogonality condition (22), has been proved in reference [8]. Appearing in equation (25),

$$\mathbf{u}_0 = [w_0, \psi_0, \gamma_0, z_{10}, \ldots, z_{r0}, \varphi_{10}, \ldots, \varphi_{r0}]^{\mathrm{T}} \text{ and } \dot{\mathbf{u}}_0 = [\dot{w}_0, \dot{\psi}_0, \dot{\gamma}_0, \dot{z}_{10}, \ldots, \dot{z}_{r0}, \dot{\varphi}_{10}, \ldots, \dot{\varphi}_{r0}]^{\mathrm{T}}$$

are vectors of initial displacement and velocity, respectively, according to initial conditions (3) presented in the scalar form.

An effective calculation of C_n is possible after transforming formula (25) into the scalar form

$$C_n = J_n / N_n, \tag{26}$$

where

$$J_n = \int_0^l \left[\mu(\mathrm{i}\boldsymbol{\nu}_n w_0 + \dot{w}_0) W_n + \mu^*(\mathrm{i}\boldsymbol{\nu}_n \psi_0 + \dot{\psi}_0)\Psi_n + \alpha D \gamma_0 \Gamma_n + \beta R \frac{\mathrm{d}\psi_0}{\mathrm{d}x}\frac{\mathrm{d}\Psi_n}{\mathrm{d}x} \right] \mathrm{d}x$$

$$+ \sum_{j=1}^{r} [m_j(\mathrm{i}\boldsymbol{\nu}_n z_{0j} + \dot{z}_{0j}) Z_{jn} + m_j^*(\mathrm{i}\boldsymbol{\nu}_n \varphi_{0j} + \dot{\varphi}_{0j})\Phi_{jn}$$

$$+ c_j(w_{0j} - z_{0j})(W_{jn} - Z_{jn}) + c_j^*(\psi_{0j} - \varphi_{0j})(\Psi_{jn} - \Phi_{jn})],$$

$$N_n = \int_0^l \left[2\mathrm{i}\boldsymbol{\nu}_n(\mu W_n^2 + \mu^* \Psi_n^2) + \alpha D \Gamma_n^2 + \beta R \left(\frac{\mathrm{d}\Psi_n}{\mathrm{d}x}\right)^2 \right] \mathrm{d}x$$

$$+ \sum_{j=1}^{r} [2\mathrm{i}\boldsymbol{\nu}_n(m_j Z_{jn}^2 + m_j^* \Phi_{jn}^2) + c_j(W_{jn} - Z_{jn})^2 + c_j^*(\Psi_{jn} - \Phi_{jn})^2]. \tag{27}$$

Now by using the results obtained for eigenfrequencies $\boldsymbol{\nu}_n$, the constants C_n and components of eigenvectors $\mathbf{U}_n$, one can transform equation (24) into the scalar form

$$\begin{bmatrix} w \\ \psi \\ \gamma \\ z_j \\ \varphi_j \end{bmatrix} = \sum_{n=1}^{\infty} C_n \begin{bmatrix} W_n \\ \Psi_n \\ \Gamma_n \\ Z_{jn} \\ \Phi_{jn} \end{bmatrix} \exp(\mathrm{i}\boldsymbol{\nu}_n t), \quad j = 1, 2, \ldots, r. \tag{28}$$

Alternatively, by transposing the complex components appearing on the right-hand side of equation (28) into trigonometrical form and because of the existence of complex conjugated components, solution (28) takes a more classical explicit form

$$w(x,t) = \sum_{n=1}^{\infty} |C_n\|W_n| e^{-h_n t}\cos(\omega_n t + \Theta_n + \operatorname{Arg} W_n),$$

$$\psi(x,t) = \sum_{n=1}^{\infty} |C_n\|\Psi_n| e^{-h_n t}\cos(\omega_n t + \Theta_n + \operatorname{Arg} \Psi_n),$$

$$\gamma(x,t) = \sum_{n=1}^{\infty} |C_n\|\Gamma_n| e^{-h_n t}\cos(\omega_n t + \Theta_n + \operatorname{Arg} \Gamma_n),$$

$$z_j(t) = \sum_{n=1}^{\infty} |C_n\|Z_{jn}| e^{-h_n t}\cos(\omega_n t + \Theta_n + \operatorname{Arg} Z_{jn}),$$

$$\varphi_j(t) = \sum_{n=1}^{\infty} |C_n\|\Phi_{jn}| e^{-h_n t}\cos(\omega_n t + \Theta_n + \operatorname{Arg} \Phi_{jn}), \tag{29}$$

where $\Theta_n = \operatorname{Arg} C_n$. Finally, it is also possible to write these functions in dual pseudo-interference form as

$$\begin{bmatrix} w \\ \psi \\ \gamma \\ z_j \\ \varphi_j \end{bmatrix} = \sum_{n=1}^{\infty} |C_n| e^{-h_n t} \left[\operatorname{Re} \begin{bmatrix} W_n \\ \Psi_n \\ \Gamma_n \\ Z_{jn} \\ \Phi_{jn} \end{bmatrix} \cos(\omega_n t + \Theta_n) + \operatorname{Im} \begin{bmatrix} W_n \\ \Psi_n \\ \Gamma_n \\ Z_{jn} \\ \Phi_{jn} \end{bmatrix} \sin(\omega_n t + \Theta_n) \right]. \tag{30}$$

5. EXAMPLE

In order to test the method described, the mechanical system as shown in Figure 3 is assumed. The continuous subsystem of this combined system is an viscoelastic Timoshenko beam simply supported at the end A and fixed to the weightless flexional spring at the end B. The second end of the flexional spring is clamped onto the rigid wall at the point D. The flexural rigidity and length of the flexional spring are denoted by R_s and l_s respectively. The discrete subsystem is composed of translational and rotational oscillators, which are connected to the beam at the point C by viscoelastic constraints. In Table 1, the coefficients of the constraints of the beam ends are shown (Figure 2).

In the example presented functions (13) take the particular forms

$$W = A_1 \sin\lambda_1 x + A_2 \sin\lambda_2 x + [\kappa_{\mathrm{I}1} W_1 g_{11} + \kappa_{\mathrm{I}1}^* \Psi_1 g_{21}]\mathrm{H}(x - x_1),$$

$$\Phi = a_1 A_1 \cos\lambda_1 x + a_2 A_2 \cos\lambda_2 x + [\kappa_{\mathrm{I}1} W_1 g_{12} + \kappa_{\mathrm{I}1}^* \Psi_1 g_{22}]\mathrm{H}(x - x_1),$$

$$\Gamma = (\lambda_1 - a_1)A_1 \cos\lambda_1 x + (\lambda_2 - a_2)A_2 \cos\lambda_2 x$$
$$+ \left\{\frac{\mathrm{d}}{\mathrm{d}x}[\kappa_{\mathrm{I}j} W_1 g_{11} + \kappa_{\mathrm{I}1}^* \Psi_1 g_{21}] - [\kappa_{\mathrm{I}1} W_1 g_{12} + \kappa_{\mathrm{I}1}^* \Psi_1 g_{22}]\right\}\mathrm{H}(x - x_1). \tag{31}$$

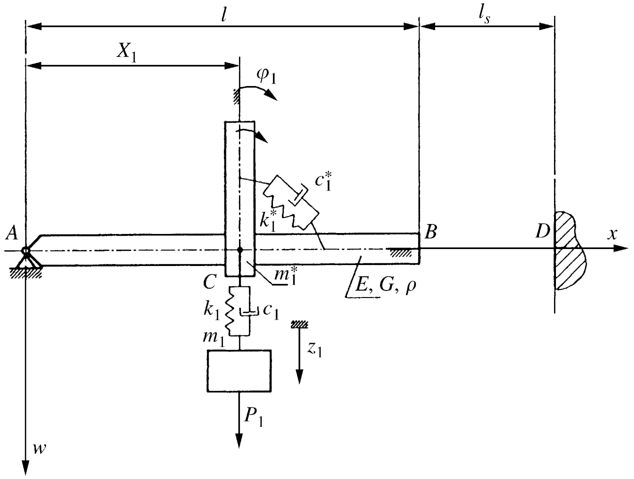

Figure 3. Exemplary Timoshenko–Kelvin beam with oscillator.

Table 1

Constraint coefficients for the beam ends of the model illustrated in Figure 2

	P	$S=1$	$S=2$
$\alpha_{sp}^{(o)}$	1	1	0
	2	0	0
$\beta_{sp}^{(o)}$	1	0	0
	2	0	1
$\alpha_{sp}^{(l)}$	1	1	0
	2	$-(1+\mathrm{i}\nu\alpha)Dl_s^3/3R_s$	$(1+\mathrm{i}\nu\alpha)Dl_s^2/2R_s$
$\beta_{sp}^{(l)}$	1	$(1+\mathrm{i}\nu\alpha)Dl_s^3/3R_s$	$1-(1+\mathrm{i}\nu\alpha)Dl_s^2/2R_s$
	2	$(1+\mathrm{i}\nu\beta)Rl_s^2/2R_s$	$(1+\mathrm{i}\nu\beta)Rl_s/R_s$

From formulas (30),

$$W_1 = A_1 \sin\lambda_1 x_1 + A_2 \sin\lambda_2 x_1, \qquad \Psi_1 = a_1 A_1 \cos\lambda_1 x_1 + a_2 A_2 \cos\lambda_2 x_1.$$

Moreover, the Green functions are the same as previously shown in equation (15), for $j=1$. Further considerations in this example have been carried out according to the algorithm presented, but for $j=1$.

6. NUMERICAL RESULTS

For numerical calculations, the data chosen for the physical quantities of the mechanical system shown in Figure 2 are: $R = 8\times10^5\,\mathrm{N\,m^2}$, $\mu = 16\,\mathrm{kg/m}$, $m = 30\,\mathrm{kg}$, $m^* = 4\,\mathrm{kg\,m^2}$, $k = 2{\cdot}5\times10^5\,\mathrm{N/m}$, $k^* = 1\times10^5\,\mathrm{N\,m}$, $c = 5\times10^2\,\mathrm{N\,s/m}$, $c^* = 1\times10^2\,\mathrm{N\,sm}$, $l = 1{\cdot}5\,\mathrm{m}$, $x_1 = 0{\cdot}75\,\mathrm{m}$, $\alpha = 0{\cdot}0005\,\mathrm{s}$, $l_s = 0{\cdot}5\,\mathrm{m}$, $R_s = 5\times10^5\,\mathrm{N\,m^2}$.

Figure 4 illustrates graphically the results obtained for certain complex eigenmodes of the mechanical system depicted in Figure 2. The first consideration is that of the phenomenon of splitting complex eigenmodes into subeigenmode pairs, i.e., their real and imaginary parts.

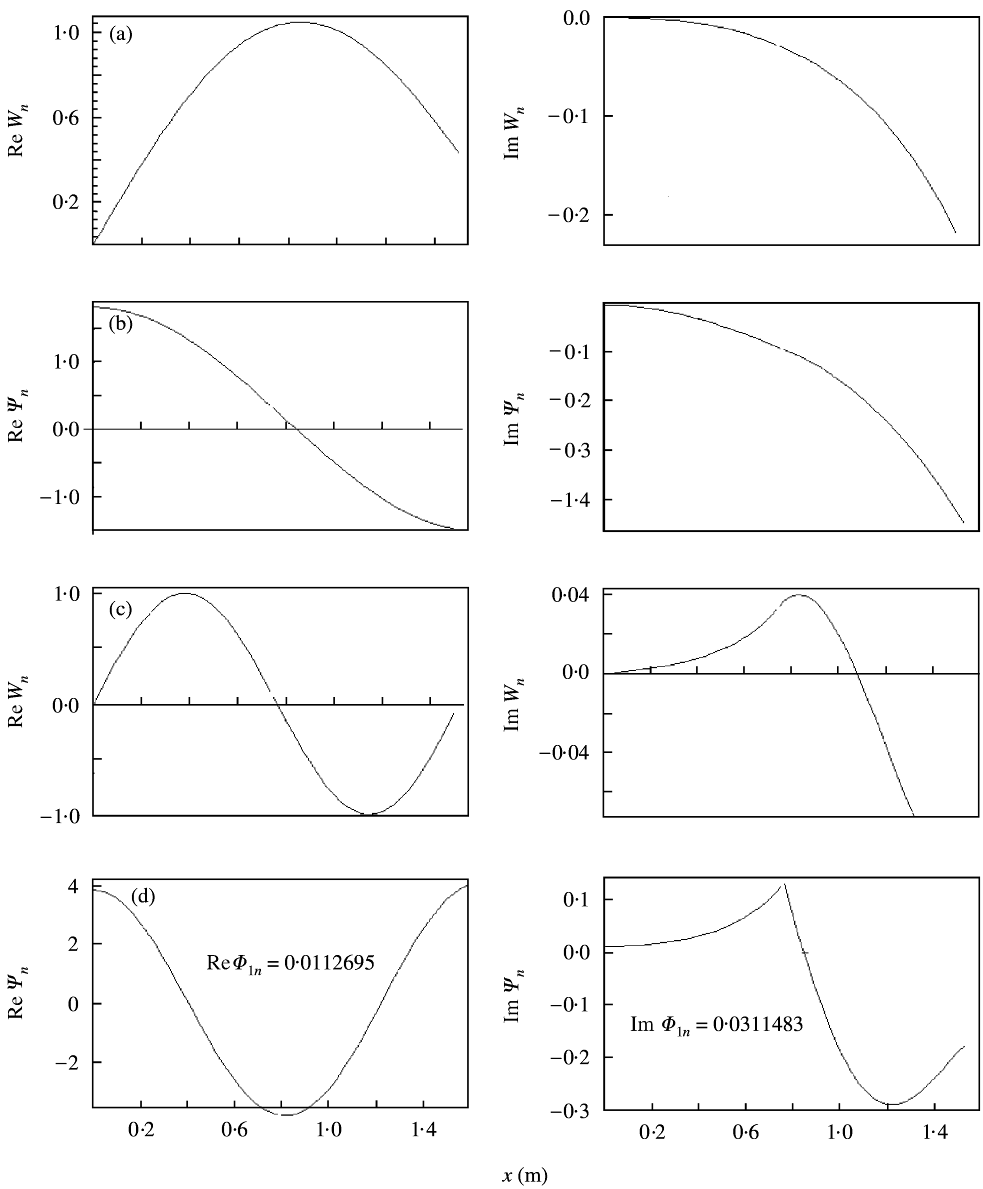

Figure 4. Complex eigenmodes W_n, Ψ_n and Z_n, Φ_n of the beam and oscillator respectively, of exemplary mechanical system shown in Figure 3. (a) Re $Z_{1n} = -$ 0·0205915. Im Z_{1n} = 0·000466203; (b) Re $\Phi_{1n} = -$ 0·0254296, Im $\Phi_{1n} = -$ 0·00801219; (c) Re Z_{1n} = 0·000642617, Im $Z_{1n} = -$ 0·00228196; (d) Re Φ_{1n} = 0·0112695, Im Φ_{1n} = 0·0311483.

From observation of the graphs shown in Figures 4(a–d) it is noticable that the real parts of each complex eigenmode are of classical regular oscillatory shapes, whilst the imaginary parts are non-regular and monotonically increase or decrease the shapes. In particular, the rotational node occurring in the imaginary part of the complex eigenmodes ψ_n (Figure 4(d)), implies that the considerable external concentrated moment produced by the inertia of the rotary oscillator acts at this point. Examples of the relationship between some

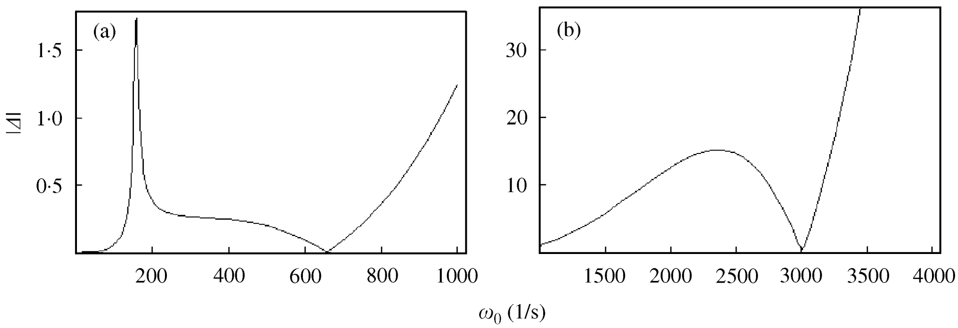

Figure 5. Graphical illustration of the first and second complex eigenfrequencies of the Timoshenko–Kelvin beam with oscillator shown in Figure 3: (a) $\mathbf{v}_1 = 18i \pm 650$, $\eta = 18$; (b) $\mathbf{v}_2 = 0{\cdot}0004i \pm 3000$, $\eta = 0{\cdot}0004$.

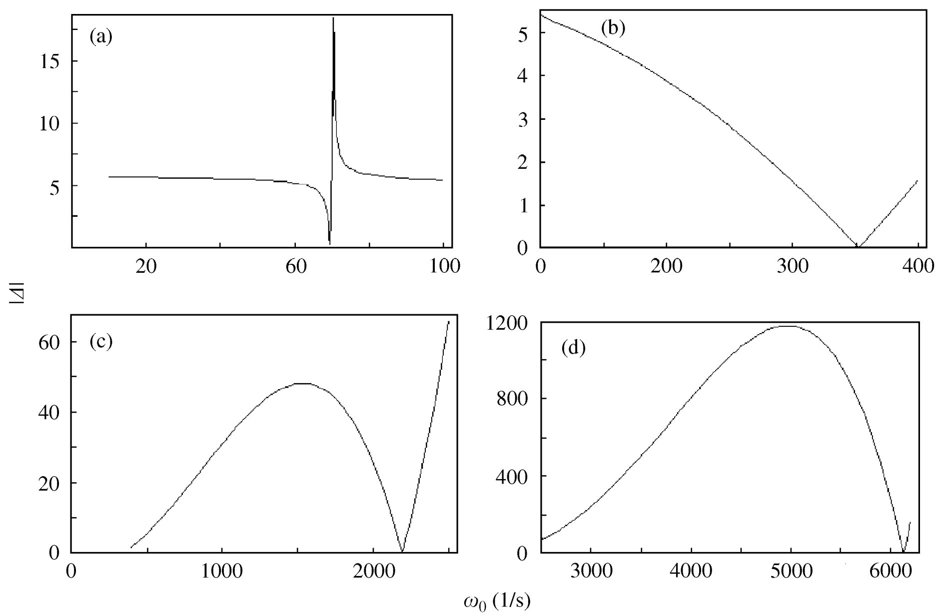

Figure 6. Graphical determination of four complex eigenfrequencies of the Bernoulli–Euler beam with oscillator shown in Figure A1: (a) $\mathbf{v}_1 = 3{\cdot}3857i \pm 69{\cdot}9$, $\eta = 3{\cdot}3857$; (b) $\mathbf{v}_2 = 1{\cdot}7576i \pm 354{\cdot}9$, $\eta = 1{\cdot}7576$; (c) $\mathbf{v}_3 = 7{\cdot}2233i \pm 2205{\cdot}7$, $\eta = 7{\cdot}2233$; (d) $\mathbf{v}_4 = 0{\cdot}0050i \pm 6169{\cdot}7$, $\eta = 0{\cdot}005$.

subeigenmodes of the beam and subeigenmodes of the oscillator corresponding to them can be seen in Figure 4. Singularities at the point of location of the oscillator appear only with the derivatives of the higher-ranked subeigenmodes. Figure 5 shows the first and second complex eigenvalues when the coefficients of the magnitude of matrix **A** determinant is zero.

A comparison of the present results with those of reference [6] is shown in Appendix A.

By defining the magnitude of the determinant **A**, in Appendix A, as the transcedental function $|\Delta|$, complex eigenvalue roots of equation (A13) (see Figure 6) occur when $|\Delta|$ is zero. The comparison of the complex eigenvalues obtained in Appendix A with the complex characteristic values reported in the first column of Table 1 of reference [6] indicates clearly that both results are identical.

Although it would appear that matrix **A** occurring in Appendix A is analytically ill-conditioned due to the singularity of the factor γ, nevertheless, the result at the neighbourhood of the singularity point (the peak in Figure 6) has been found quite precisely.

7. CONCLUSIONS

1. The generalized method presented in this paper can be applied to the free vibration analysis of a discrete–continuous system with internal damping for an arbitrary choice of parameters: α, β, c_j, c_j^*, k_j and k_j^*, and requires that self-adjoint operators **K** and **L** must not be homothetic.

2. The final results presented in the forms of equations (28)–(30) can be applied in practical calculations. The form of equation (28) indicates the utilitarian significance, whereas the forms of equations (29) and (30) the cognitive importance.

3. The form of equation (29) confirms unambiguously the correctness of this method. Moreover, the form of equation (30) indicates that the phenomenon of free vibrations is of binary character, this meaning dual interference exists. One can formulate a hypothesis that the apparent inertia, i.e., real inertia and viscosity are the reason for this dual phenomenon. This supposition is also confirmed by the analogy of kinetic and dissipative energy.

4. The operational principle describing the generalized property of complex eigenvector orthogonality (22) and operational formula (25) are invariants of this method. The results from the separation of variables (equations (8) and (9)), as well as the presence of invariants can be used for solving the free vibrations problem for any linear non-conservative system.

5. In particular cases, when operators **K** and **L** are homothetic, phase angles $\operatorname{Arg} W_n$, $\operatorname{Arg} \Psi_n$, $\operatorname{Arg} \Gamma_n$, $\operatorname{Arg} Z_{jn}$ and $\operatorname{Arg} \Phi_{jn}$ are invariable which implies that equation (29) can be reduced to the well-known form which can be obtained by the classical Fourier method.

6. Oscillators connected to the continuous subsystem often play the role of dynamic dampers or exciters of a chosen component of the subsystem vibrations. Parameters of these dampers or exciters can be easily determined by using formulas (19) and (14).

7. Solution of the steady forced-vibration problem needs

(a) completion of the right-hand side of equations (10) by providing amplitudes of forced loads:
(b) replacement of the complex frequency ν by the real frequencies of forced loads ω:
(c) the assumption that the forced frequencies of loads are the same and invariable in time.

The problem of vibration forced by arbitrary loads can be solved using the principles given by reference [8].

REFERENCES

1. H. ABRAMOWICH 1992 *Journal of Sound and Vibration* **157**, 183–189. Natural frequencies of Timoshenko beams under compressive axial loads.
2. T. P. CHANG and C. Y. CHANG 1998 *International Journal of Solids and Structures* **35**, 383–401. Vibration analysis of beams with a two degree-of-freedom spring–mass system.
3. S. H. FARGHALY and M. G. SHEBL 1995 *Journal of Sound and Vibration* **180**, 205–227. Exact frequency and mode shape formulae for studying vibration and stability of Timoshenko beam system.
4. S. KUKLA 1997 *Journal of Sound and Vibration* **205**, 355–363. Application of Green functions in frequency analysis of Timoshenko beams with oscillators.

5. M. W. D. WHITE and G. R. HEPLER 1995 *Journal of Applied Mechanics* **62**, 193–199. Vibration modes and frequencies of Timoshenko beams with attached rigid bodies.
6. M. GÜRGÖZE 1999 *Journal of Sound and Vibration* **223**, 666–677. Alternative formulations of the characteristic equation of a Bernoulli–Euler beam to which several viscously damped spring–mass system are attached in-span.
7. K. CABAŃSKA-PŁACZKIEWICZ 1999 *Engineering Transactions* **47**, 21–37. Free vibration of the system of two Timoshenko beams coupled by a viscoelastic interlayer.
8. J. CABAŃSKI 2000 *Journal of Technical Physics* **41**, 471–481. Generalized exact method of analysis of free and forced oscillations in the non-conservative physical system.
9. J. CABAŃSKI 2001 *Journal of Theoretical and Applied Mechanics* **1**, 39, 195–212. Analysis of free vibrations of a continuous–discrete system with damping.
10. J. CABAŃSKI 1993 *Zeitschrift für Angewandte und Mechanik* **73**, 148–150. Method of calculation of basic parameters of vibration eliminators by complex functions.

APPENDIX A: THE COMPARATIVE EXAMPLE

In the order to verify the method presented in this work, the study is compared with the results obtained in the example of reference [6]. The continuous–discrete system shown in Figure A1 is assumed. The continuous subsystem of this combined system is a clamped–free Bernoulli–Euler beam. Moreover, the discrete subsystem is a translational oscillator, i.e., a discrete rigid body (particle), which is connected to the beam by means of a viscoelastic constraint described by the Voigt–Kelvin model.

The governing equations of the oscillating motion of the system shown in Figure A1, are

$$EJ\frac{\partial^4 w}{\partial x^4} + \mu\frac{\partial^2 w}{\partial t^2} + \left(k + c\frac{\partial}{\partial t}\right)(w - z)\,\delta(x - x_1) = 0,$$

$$[m\,\mathrm{d}^2 z/\mathrm{d}t^2 - (k + c\,\mathrm{d}/\mathrm{d}t)(w_1 - z)]\,\delta(x - x_1) = 0, \tag{A1}$$

which together with the assigned boundary conditions

$$w(0,t) = 0, \qquad \partial w/\partial x|_{x=0} = 0, \qquad EJ\,\partial^2 w/\partial x^2|_{x=l} = 0, \qquad EJ\,\partial^3 w/\partial x^3|_{x=l} = 0 \tag{A2}$$

represent the boundary-value problem.

By comparison of equations (A1) with the vector equation (6), one can identify the vector of displacements as

$$\mathbf{u} = \begin{bmatrix} w \\ z \end{bmatrix} \tag{A3}$$

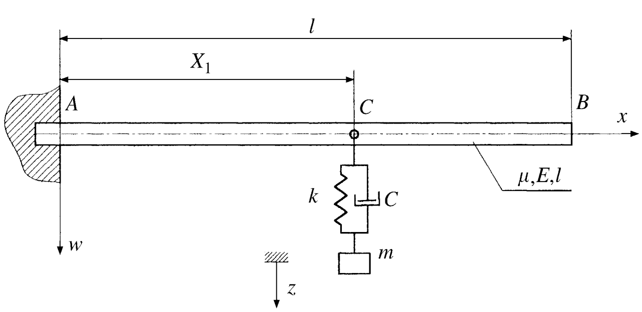

Figure A1. Clamped–free Bernoulli–Euler beam with oscillator.

and operators of inertia **M**, damping **L** and stiffness **K**, respectively, as

$$\mathbf{M} = \begin{bmatrix} \mu & 0 \\ 0 & m\delta(x - x_1) \end{bmatrix}, \quad \mathbf{L} = c\begin{bmatrix} \delta(x - x_1) & -\delta(x - x_1) \\ -\delta(x - x_1) & \delta(x - x_1) \end{bmatrix},$$

$$\mathbf{K} = k\begin{bmatrix} [(R/k)\partial^4/\partial x^4 + \delta(x - x_1)] & -\delta(x - x_1) \\ -\delta(x - x_1) & \delta(x - x_1) \end{bmatrix}. \tag{A4}$$

The above note implies that these operators are self-adjoint, but they are not homothetic.

After separation of the variables in equations (A1), according to the procedure used in equations (6–9), the set of one ordinary differential equation and a linear algebraic one can be written as

$$\mathrm{d}^4W/\mathrm{d}x^4 - \lambda^4 W + \gamma W_1 \delta(x - x_1) = 0, \qquad Z = -\kappa W_1, \tag{A5}$$

where

$$\lambda = \sqrt[4]{\frac{\mathbf{v}^2\mu}{EJ}}, \quad \gamma = \frac{\mathbf{v}^2 m(k + \mathrm{i}\mathbf{v}c)}{[\mathbf{v}^2 m - (k + \mathrm{i}\mathbf{v}c)]EJ}, \quad \kappa = \frac{k + \mathrm{i}\mathbf{v}c}{\mathbf{v}^2 m - (k + \mathrm{i}\mathbf{v}c)}. \tag{A6}$$

In conformity with separation of the variables, the boundary conditions (A2) can be written down in the form

$$W(o) = 0, \qquad \mathrm{d}W/\mathrm{d}x|_{x=0} = 0, \qquad \mathrm{d}^2W/\mathrm{d}x^2|_{x=l} = 0, \qquad \mathrm{d}^3W/\mathrm{d}x^3|_{x=l} = 0. \tag{A7}$$

The general solution of the differential equation in equation (A5) has the form

$$W = A\sinh\lambda x + A^*\sin\lambda x + B\cosh\lambda x + B^*\cos\lambda x$$
$$- (\gamma W_1/2\lambda^3)\,[\sinh\lambda(x - x_1) - \sin\lambda(x - x_1)]\mathrm{H}(x - x_1). \tag{A8}$$

Substituting function (A8) and its first distributional derivative into two first (upper) boundary conditions (A7), respectively, one concludes that $A^* = -A$ and $B^* = -B$, from whence equation (A8) reduces to the form

$$W = A(\sinh\lambda x - \sin\lambda x) + B(\cosh\lambda x - \cos\lambda x)$$
$$- (W_1\gamma/2\lambda^3)[\sinh\lambda(x - x_1) - \sin\lambda(x - x_1)]\mathrm{H}(x - x_1). \tag{A9}$$

From equation (A9)

$$W_1 = A(\sinh\lambda x_1 - \sin\lambda x_1) + B(\cosh\lambda x_1 - \cos\lambda x_1). \tag{A10}$$

Owing to the second and third distributional derivatives of the piecewise regular function (A9), two other (lower) boundary conditions (A7) together with associated equation (A10) constitute the system of homogeneous linear algebraic equations,

$$\mathbf{AX} = \mathbf{0}, \tag{A11}$$

where

$$\mathbf{X} = [A, B, W_1]^{\mathrm{T}},$$

$$\mathbf{A} = \begin{bmatrix} (\sinh\lambda l + \sin\lambda l) & (\cosh\lambda l + \cos\lambda l) & -(\gamma/2\lambda^3)[\sinh\lambda(l - x_1) + \sin\lambda(l - x_1)] \\ (\cosh\lambda l + \cos\lambda l) & (\sinh\lambda l - \sin\lambda l) & -(\gamma/2\lambda^3)[\cosh\lambda(l - x_1) + \cos\lambda(l - x_1)] \\ (\sinh\lambda x_1 - \sin\lambda x_1) & (\cosh\lambda x_1 - \cos\lambda x_1) & -1 \end{bmatrix}. \tag{A12}$$

For the existence of non-trival solutions of equation (A11), the determinant of the coefficient matrix $\mathbf{A}$ with respect to v must vanish, which leads to the exact characteristic equation:

$$\det \mathbf{A} = 0. \tag{A13}$$

The data for the example calculated here were $EJ = 6 \times 10^5$ N m^2, $\mu = 12$ kg/m, $l = 1{\cdot}5$ m, $x_1 = 0{\cdot}75$ m, together with the parameters used examples from reference [6], i.e., $m/\mu l = 1$, $kl^3/EJ = 0{\cdot}5$, $(c/2m)\sqrt{m/k} = 0{\cdot}05$, from which it follows directly that $m = \mu l$, $k = 0{\cdot}5\,EJ/l^3$, $c = 0{\cdot}1\sqrt{mk}$. The relationship between the complex eigenfrequencies $\mathbf{v}_n$ and the non-dimensional complex characteristic values λ_n^* here presented in reference [6] is

$$\lambda_n^* = \eta^* \mathbf{v}_n \quad (n = 1, 2, \dots, \infty),$$

where $\eta^* = \mathrm{i}\sqrt{\mu l^4/EJ}$ is the conversion factor.

APPENDIX B: NOMENCLATURE

l	beam length
t	time
x	Cartesian co-ordinate system axis
x_j	co-ordinate of location of jth oscillator
ρ	mass density of the beam material
E, G	Young's and Kirchoff's modulus of the beam material respectively
α, β	longitudinal and tangential retardation times of the beam material respectively
k'	shearing factor
m_j	mass of the jth oscillator corresponding to the particle
m_j^*	mass moment of inertia of disc of jth oscillator with respect to its axis of rotation
k_j, k_j^*	stiffness coefficients of translational and rotational constraints of the jth oscillator respectively
c_j, c_j^*	damping coefficients of translational and rotational constraints of the jth oscillator respectively
K_A, K_B	general stiffness coefficients of the beam constraints at its ends respectively
w	deflection of the beam, $w = w(x, t)$
ψ	angle of rotation of the beam cross-section, $\psi = \psi(x, t)$
γ	angle of shearing of the beam cross-section, $\gamma = \gamma(x, t)$
z_j	displacement of the particle of the jth oscillator, $z_j = z_j(t)$
φ_j	angle of rotation of the disc of the jth oscillator, $\varphi_j = \varphi(t)$
A	cross-section area of the beam
I	axial moment of inertia of the beam cross-section
q	distributed force acting on the beam, $q = q(x, t)$
m	distributed moment acting on the beam, $m = m(x, t)$
Q_j	concentrated force acting on the jth oscillator, $Q_j = Q_j(t)$
M_j	concentrated moment acting on the jth oscillator, $M_j = M_j(t)$